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Notes on the First Chapter of *The Continuum*: Intension, Extension, and Arithmetism

Julien Bernard

The particular position of *The Continuum* among the constructivist mathematics of the twentieth century

1 In 1918, *The Continuum* was published, the first and greatest book of Hermann Weyl about the foundations of mathematics. The position he defended is one of the first members of a wide family of approaches of mathematics we call now “constructivism”, which has been developed since the beginning of the twentieth century. Those atypical ways of doing mathematics involve restricting the mathematical processes to constructive ones. We can give several senses to this notion of “construction”. In particular, we can restrict the *universe of mathematical* objects (to only objects that can be, in some sense, “constructed”), or restrict the mathematical demonstrations in order to make the notion of mathematical existence coincide with the notion of “being constructible”.

2 Those kinds of approach were first regarded as too restrictive. Indeed, being constructivist in the beginning of the twentieth century compelled us to abandon a large part of the most powerful results of mathematics. This situation has changed since the rapid expansion of constructivist mathematics from the end of the 1960’s. The works of mathematicians like Errett Bishop and Per Martin Löf spurred faster growth of constructivist reconstructions of mathematics. This expansion still continues. Therefore, we no longer wonder if the constructivist approach is able to construct powerful mathematics but rather if we have good reasons to accept the restriction of our mathematical processes to constructive ones.

3 The theory of recursive functions and the development of computer science have given favourable arguments for the search of constructive mathematics with a domain of

study that coincides with the effectively computable objects (functions, numbers, operations and sets). This is a strong sense we can give to the notion of "construction". Nevertheless, we have to remember that the original ideas that founded the constructive approaches were those of mathematicians who sought sound foundations for coherent and meaningful mathematics. That's why, in order to understand the motives that guide the development of constructive mathematics, we have to review the works of those pioneers who conceived the constructive approach as a trial to answer the problem of the foundations of mathematics.

4 Luitzen Egbertus Jan Brouwer is regarded as the first mathematician to have systematically developed such an approach. For Brouwer, mathematics is a free creation of the human mind. His personal approach on constructive mathematics, usually called "intuitionism", was therefore guided by the internal nature of mathematics. Historians and philosophers of science have remembered his work as the beginning of constructivist approaches because of its great scope. Indeed, Brouwer had developed his intuitionist position throughout his life. Nevertheless, the success of this author has partly concealed other previous constructivist approaches, which were based on noticeably different positions on philosophy of science. *The Continuum* is one of the most important of those first attempts.

5 We could think that constructivist mathematics could be studied without reading such a work. Indeed, Hermann Weyl himself said at the beginning of the 1920's that he joined the intuitionist position of Brouwer, which was a more daring criticism to the spirit of Set Theory than his own. Should we conclude that *The Continuum* is just a kind of sketch of the completed constructivist position: that of Brouwer? We shouldn't. In spite of the similarity between the two approaches, which justifies that they both belong to the wide family of constructivism, the methods and motives of the two authors are rather different. First, the constructivism of Brouwer calls the Excluded Middle Principle (and so classical logic) into question, whereas Hermann Weyl wants to give consistency to mathematics in maintaining the Excluded Middle Principle by restricting the universe of mathematical entities. We can explain this difference because L.E.J. Brouwer more violently criticizes the usual notion of infinity in mathematics than Hermann Weyl. For L.E.J Brouwer, arithmetic (of natural numbers) makes a bad use of this notion as well as analysis (of real numbers) (for example [Weyl 1949, 150–151]). For him, a proposition that speaks about the totality of natural numbers is nonsense if there is no constructive prove of it or any counterexample. When he understood the work of L.E.J Brouwer, Hermann Weyl admitted that we have no more intuition about the totality of natural numbers than about real numbers. On this particular point, the position of *The Continuum* was not radical enough. There was a last residue of the Platonic position about mathematics.

6 But we can see a more profound difference in the motives of the two thinkers, a difference into the nature of the philosophical thoughts that are involved. Indeed, the philosophy of science developed in *The Continuum* is not purely internal to mathematics like Brouwer's one. Hermann Weyl tries to provide a type of mathematics that is required for the construction of physics. *The Continuum* is therefore only a part of an ambitious epistemological program about natural science and its link with mathematics. It is significant that the main work of Hermann Weyl about the general theory of relativity, *Space-Time-Matter*, is published the same year as *The Continuum*. Hermann Weyl thinks that we can't construct a correct philosophy of mathematics

without knowing the interactions between mathematics and physics. The originality and ambition of this epistemological program makes the study of *The Continuum* still interesting. It gives us new motive to accept a kind of (moderate) constructive position about mathematics, in order to be able to set up strong interactions between mathematics and physics.

The Continuum Problem, a Bridge between the Philosophy of Mathematics and the Philosophy of Physics of Hermann Weyl

7 In *The Continuum*, Hermann Weyl defended his position on philosophy of science by focussing on the continuum problem (a problem involving simultaneously mathematics, physics and phenomenology of perception). We can begin examining this problem by noting that the intuitive and the mathematical continua do not coincide. The *intuitive continuum* is given immediately by our perceptive intuitions of time and space extension. "Immediately" means that we set at a phenomenal level, before any conceptual reconstruction and, in particular, before the physical one. The objects given by this perceptive intuition are continuous in that their parts are closely related and are themselves new continuous objects. The whole is before the parts. At this perceptive level, the operation of division never leads us to something like "indivisible points" that would constitute the continuous object. On the contrary, the *mathematical continuum*, as we understand it since the development of mathematical analysis, is actually *constituted* by individual isolated points: the *real numbers*. Hermann Weyl calls this mathematical conception of the continuum "atomistic" as opposed to the intuitive one.

8 In accordance with Hermann Weyl's general epistemological program, his answer to the continuum problem is directed toward the possibility of providing foundations for physics. For Weyl, in 1918, the atomistic feature of the mathematical continuum does not imply that analysis failed in its task. The mathematical continuum has to be atomistic because we must construct it *arithmetically*, beginning our construction at the discrete infinite sequence of the natural numbers that is the very foundation of mathematical thought [Weyl 1918, 48]. There is an insurmountable gap between analysis and the perceptive intuition. To solve the continuum problem, we have to assume this gap and therefore to assume the atomistic position inherent to mathematical analysis. Analysis is not a phenomenology of perception but a *theory of the Continuum*. In this way, it must be ultimately justified by its insertion in the whole theory of physics [Weyl 1918, 93].

9 To construct this arithmetical continuum, two types of objects are then relevant: *sets and functions*. They have to be introduced in order to express the physical continuity of space and motions. All the mathematical and logical work accomplished by Hermann Weyl in the first chapter of *The Continuum* could be regarded as an analysis of the notion of function (and of the notion of set that is a particular case of the first, for Hermann Weyl, as we will see). Following Hermann Weyl, we should remark that, in the history of mathematics and physics, we find a first way to think about a function. It is derived from algebra. According to this view, a function is a formula obtained by the iterated applications of the four usual algebraic operations ($+, *, -, /$) and of some

transcendental elementary functions (*sin*, *exp*, *etc.*) This position is insufficient because this notion of function is not enough extended to express all the functions needed by modern analysis and physics [Weyl 1918, 45–47]. At the opposite side, we have another conception of function that is very general and quite vague. According to this, a function is thought of as a pure correspondence between two domains of objects. This correspondence is pure in the sense that it is thought of independently of the existence of an explicit relation that links together the elements of those domains. This conception is often attributed historically to Dirichlet [Weyl 1918, 23]. For Weyl, this notion of function is nonsensical and can't have any physical application.

10 In order to avoid these difficulties, the first chapter of *The Continuum* gives new bases to the notions of sets and functions. To be useful for physics, a function has to be thought as an *intelligible* relation. That's why Weyl assumes that a set or a function can't have any sense if it is not linked to a law; i.e. an explicitly constructed relation. But, in order to give all its extension to the notion of relation, Weyl proposes to replace the elementary algebraic operations by some general logical principles. This choice of restricting the mathematical functions to those which can be defined by a logically constructed relation is the main reason why we can think about the position of Hermann Weyl in *The Continuum* as a member of the family of constructivism.

The Obstacles to the Intelligibility of the First Chapter of *The Continuum*

11 The reading of the first chapter of *The Continuum* is a difficult exercise because of the unusual symbolism used by Weyl, because of the distance with Set Theory with which we are used to, and because of Weyl's choice not to present his final system directly but all the previous steps also. (In his introduction to the English translation, Stephen Pollard expresses clearly those difficulties. [Weyl 1918, xv]). This paper is a contribution toward the interpretation of Weyl's intentions in the first chapter of *The Continuum*.

12 We aim to clarify this matter, using other Weyl papers of the same period or later, and measuring the differences that separate Weyl's notions of sets and functions from those we use now. We don't make a complete reconstruction of Weyl's system in a modern fashion, such approaches have been already proposed (cf. [Feferman 1988] and [Feferman 1997]), but rather focus on some central problems in the nature of natural numbers and the shift Weyl is obliged to apply to the traditional distinction between intension and extension.

13 We have already stressed that, in order to give to analysis an entire intelligibility, and applicability to physics, Weyl has to impose some restrictions on the entities admitted in the mathematical universe. Before specifying the exact interpretative problems with which we are concerned, let's give a quick presentation of the main principles of restriction he adopts. They can be sketched by four terms: definitionism, intuitionism, predicativism, and arithmetism. Let's explain what those terms mean.

Restrictive Principles on Entities Adopted in *The Continuum*

Definitionism

14 This position refuses to assume a new ideal object (that is a set or a function) if it is not introduced by the way of an explicit definition of the relation that links its constituent elements together. More precisely, an explicitly given relation (or property) is a relation constructed by the logical principles (cf. below) from the primitive entities of the mathematical domain with which we are concerned. Since the mathematical objects are always preceded by a logical construction, we can distinguish, according to the logical tradition, between an intensional level and an extensional one.

Intuitionism

15 Hermann Weyl's mathematical universe is restricted in that all the entities assumed must be generated by the logical principles from a *basic category of entities* that is given *intuitively*. This basic category is a *structure* made up of *primitive* objects and relations. The intuitive knowledge we have of those entities give foundations to the Excluded-Middle Principle. Each rightly built proposition that concerns only the primitive entities (in particular "existential propositions" which assert that *there is* one basic entity which satisfy one given property) admits one truth-value, regardless our ability to determine it. Weyl says that such a category is a "**complete** system of definite self-existent objects". (In the present paper, we will always use the term "complete" in this sense.) We mentioned above that Weyl gave up this feature of his position when he knew the more radical thought of Brouwer (cf. p. Error: Reference source not found) middle of paragraph starting with "We could think…").

Predicativism

16 Hermann Weyl refuses every *impredicative* definition, that is every definition which supposes the prior given information of a totality of entities of which the object to define is one of the members. Weyl's predicativism is expressed by the "***restricted principle***". It consists in restricting the scopes of the quantifiers[1] to the primitive entities. This principle permits elimination of the impredicative definitions while expressing the privileged access we have to the basic categories.

The Logical Principles

17 Those three major theses of Weyl's position (definitionism, intuitionism and predicativism) give rise to the formulation of 6 logical principles that can be indifferently interpreted as principles of construction of propositions or as principles of construction of relations.

18 There are six principles:

1) The negation principle
2) The blanks-identification principle
3) The conjunction principle

4) The disjunction principle
5) The “filling in” principle
6) The “there is” principle [Weyl 1918, 9–10]

19 We don’t give details of those principles. They permit construction of each property (or relation) to be expressed in the first-order predicate language, from the primitive relations, and including the equality symbol “=” for primitive objects and symbols for sets, functions and for the membership relation “∈”. According to Weyl’s restricted principle, the scopes of the quantifiers are restricted to the primitive categories.

Arithmetism

20 Weyl demands a very strong completeness condition from the basic categories. Owing to this fact, he develops only one such category: that of natural numbers. The only primitive relation is that of succession.

21 The natural numbers series is important in Weyl’s position because it makes up the intuitive datum that permits the foundation of a new type of definition of relation (the principle of iteration) and therefore a new form of inference (the inference by complete induction). Let’s call “arithmetism” the particular foundational status Weyl gives to the natural numbers series.

22 We are now ready to expose the problems with which we are concerned.

Two Problems in the Understanding of Weyl’s Thought

23 After having introduced the logical principles, Hermann Weyl adds two more principles to his system. First, the *principle of substitution* that permits a relation taken as an argument for another relation. Secondly, the *principle of iteration* that is the foundation of a kind of recursive definition for set-functions. Finally, Hermann Weyl introduces the *mathematical process* that expresses the passage from the intensional point of view of relations to the extensional point of view of sets and functions. For Hermann Weyl, this corresponds to the passage from the domain of logics to the domain of mathematics. The main difficulties in the understanding of Weyl’s thought arise here.

24 The first difficulty is linked to the distinction between intension and extension. According to his predicativist position, Weyl admits only the relations that can be defined when the *only totality* available is that of natural numbers. In so far, all the relations have to be generated before the first and only application of the mathematical process that introduces the extensional entities in the system (and in particular real numbers).

25 At first sight, this limitation to a single application of the mathematical process seems to be incompatible with two others assertions of Weyl: 1) the fact that the two last principles have to use sets and functions and 2) the fact that Weyl continues to speak of a *hierarchy* between the relations of his system. How is it possible that sets and functions, which are extensional entities, step in the construction of relations (intensions), before the only application of the mathematical process? Moreover, why does Weyl continue to speak of a hierarchy of entities in his system if there is only one application of the mathematical process?

26 In order to solve these interpretative problems, we have to make explicit the shift Weyl is obliged to apply to the usual notions of intension and extension. That will be our first task.

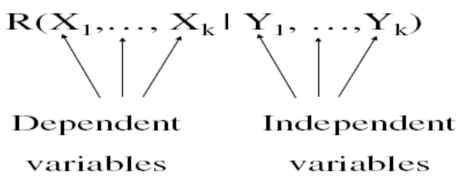
27 The second difficulty is directly linked to the principle of iteration. According to Solomon Feferman [Feferman 1988], the particular form Weyl gives to this principle is problematic because we know, since a work of Kleene, that it permits to get over the first level, in the strict sense of the Ramified Theory of Types of Bertrand Russell [Russell 1927] contrary to what Weyl himself tells. Our second task will be, not to judge if his principle of iteration really betrays Weyl's predicativist position, but rather to explain the reason for which Weyl expresses this principle in this particular form. It will call for making explicit the status Weyl is giving to the notion of natural numbers series.

The Shift of the Traditional Distinction between Intension and Extension

The Mathematical Process

28 In order to understand this shift, we have to display more precisely the mathematical process. In accordance with the traditional way of speaking, all the relations that can be constructed by the mean of Weyl's logical principles from the primitive entities will be called "intensional entities". The *mathematical process* establishes a transition between those intensional entities and the extensional entities that are sets and functions.

29 Let's assume we have an intensional relation *R*(,,...) Each argument is linked to a category that can be basic or not (for example, the category of sets of numbers). We have to suppose that the arguments are divided in two groups: the dependent ones and the independent ones.



30 (In this paper, we will always adopt Weyl's notation, using the "—" symbol to distinguish dependent variables from independent ones.)

31 • Let's talk firstly of the case n = 0, that is the case where there is no independent variable. Then, the mathematical process links to the relation $R(x_1,\ldots,x_k)$ the k dimensional set $\check{R}$ such as we have:

$R(x_1,\ldots, x_k)$ iff $(x_1,\ldots, x_k) \in \check{R}$

Two factors step in this transition from the relation to the set:

1) The variables disappear. In Gottlob Frege's way of speaking, the set $\check{R}$ is a *saturated* entity on the contrary to the relation from which it is constructed.

2) The identification criterion changes. Let's quote Weyl.

> "Therefore, how two sets [...] are defined [...] does not determinate their identity [on the contrary to *relations*]. Rather an *objective* fact, which is not decidable from

> the definition in a purely logical way, is decisive; namely, whether each element of the one set is also an element of the other, and conversely." [Weyl 1918, 20]

32 • Let's assume now that $n \neq 0$, that is there is at least one independent variable. Then, the mathematical process links to the relation $R(x_1, ..., x_k|y_1, ..., y_n)$ the ***function*** $\check{R}(y_1, ...,y_n)$ such as we have:

$R(x_1, ..., x_k|y_1, ..., y_n) \Leftrightarrow (x_1,..., x_k) \in \check{R}(y_1, ...,y_n)$

For each possible value of the arguments, the function $\check{R}(y_1, ...,y_n)$ becomes a set. This transition from the relation to the function includes two factors:

1) One part of the variables disappears (the dependent ones).

2) The identification criterion changes in a similar way to the case of sets. Two functions are identical if their values are identical for each possible determination of the variables.

33 Thus, we find the same two factors. Let's make two remarks.

34 First, we can see immediately the difference between *this* notion of function and the *set-theoretical* one to which we are used. Indeed, in Set Theory, the value of a function can be of any nature whereas in Weyl's system the value of a function must be a set, *i.e.* can't be a basic object. But this difference is not an essential one. In fact, in order to express a one-to-one relation between basic objects, the notion of *set* (in Weyl's sense) is enough. Such a relation can be rendered by a two-dimensional set $\check{R}$ that verify the property that for each x there is one y such as "$(x,y) \in \check{R}$".

35 Secondly, we can see that, in *The Continuum*, the notion of function is defined as an extension of the notion of set. The sets become *borderline* cases of functions: those where the number of independent variables has been reduced to 0.

Comparison with the Set-Theoretical Distinction between Intension and Extension

36 We have distinguished two moments within the transition from relations to sets and functions: 1) the disappearance of some variables (let's call it "*the abstraction process*") and 2) the change of identification criterion. In fact, each of those two processes have his own autonomy and this is a distinctive feature of Weyl's system. Let's compare it with Set Theory.

37 In Set Theory, intensions are generally thought as propositional functions. More precisely, they are given by formulas of the set-theoretic language (for example: the first-order predicates language + the binary relation "$\in$") with one (or several) unbounded variable(s) (of sets). In Set Theory, the principle that permits to make the transition from intensions to extensions is the "axiom of separation"[2]. According to this principle, for each given set X and for each set theoretical propositional function $\Phi(x)$, 'x' being the unbounded variable of the formula, we can assert the existence of the set $\{x \in X|\Phi(x)\}$ of all the elements of X which make true the functional proposition $\Phi(x)$.

38 On the contrary to intensions, sets are saturated entities (they don't contain any variable). Moreover, the criterion of identity between sets is extensional, *i.e.* two sets are equals if and only if they have the same elements. In so far, we can see that the process of abstraction and the transition from a logical identity to an extensional one

are simultaneous. That is the way the notions of intension and extension are usually conceived in Set Theory.

39 In *The Continuum*, Weyl regards those two processes as independent. It is never explicitly explained in *The Continuum* but Weyl gives a few indications in this direction [Weyl 1918, 40–41] and he confirms this fact in his *Letter to Hölder* [Weyl 1919, 114–117]. (Cf. also the explanation we give below of "formal" sets and functions)

40 As a result of this independence, there are several types of intensional and of extensional entities in Weyl's universe. As we saw above, when the abstraction process is entirely applied to a relation, suppressing all the variables, the relation becomes a set. And when the abstraction process is partly applied to a relation, leaving some residual variables, the relation becomes a function. Those entities can be considered at a logical level, *i.e.* before the change of identification criterion. This independence, in Weyl's system, between the abstraction process and the change of identification criterion permits to understand why we can use sets and functions in the definition of relations (intensions) in spite of the fact that all definitions of intensional entities have to be obtained before the single application of the mathematical process that generate extensional entities. Weyl tells that sets and functions are used in the definition of relations (intensions) only "in a purely formal way" [Weyl 1918, 40]. In so far, let's call "*formal*" the sets and the functions used in the definitions of relations (intensions). They are abstract entities for which the identification criterion is still a logical one. That's why, in Weyl's system, we can't use the (extensional) identity of two sets or functions in the definition of a relation. Those formal sets and functions are useful to express the fact that a relation is taken as an object for another relation and to express naturally the substitution and iteration principles (cf. below). Moreover, in his letter to Hölder, Weyl specifies that the use of formal sets and functions wouldn't have permitted to construct new relations between primitive entities without the iteration principle. [Weyl 1919, 117]

41 As a consequence of the uniqueness of the application of the mathematical process, the intensional level becomes in Weyl's system entirely independent of the extensional one. This is contrary to Set Theory where there is infinity of intensional and the extensional levels which are interconnected, extensional entities being used in defining intensions[3].

Weyl's arithmetism: iteration and natural numbers series

42 Let's deal now with the second difficulty. We have to explain Weyl's principle of iteration by analysing what we called his "arithmetism". To understand why Weyl put the natural numbers series in the centre of the foundations of mathematics, we must make implicit his conception of the notion of natural number.

43 Two types of intuition are linked to the natural numbers series:

1) We have the *intuition* that gives us the natural numbers series as a *complete system.* (We defined the word "complete" above).

2) We have the *intuition of the iteration*. We mean here the intuition by which we can assert that, when we have a homogeneous operation (that is an operation which links

each object to another object of the same category), we can then consider the iteration of this operation an indefinite number of times.

44 Those two intuitions are linked together because:

1) The category of natural numbers can be regarded as the totality of the elements obtained by the *iteration* of the "successor relation" from the number 1.
2) By the means of the intuition of iteration, the completeness of the category of natural numbers applies to every totality of ideal objects isomorphic to it. This isomorphism means not only that this totality can be enumerated (in the usual sense we give to this word in mathematics), but also that this totality is produced by an indefinite number of iterations of the same well-defined operation (for example: a recursive series of numbers).

45 Those two intuitions (that of the "completeness" of the natural numbers series, and that of the possibility to repeat indefinitely the iteration of an homogeneous operation) are blending together so that Weyl seems to identify them entirely:

> "[...] the idea of iteration, *i.e.*, of the sequence of the natural numbers, is an ultimate foundation of mathematical thought" [Weyl 1918, 48]

46 The second feature we have expounded is an essential one for Weyl's system[4]. It shows that he had, in a way, a *formal* conception of natural numbers in spite of his rejection of a *formalist* opinion on mathematics. His conception of natural numbers is a *formal* one in the sense where what is essential in the natural numbers series is its *iterative structure*. That's why Weyl think about all series isomorphic to the natural numbers as complete ones.

47 Nevertheless, his position is not a *formalist* one because he didn't think at all that this structure emerges from arbitrary choices such as the conventional acceptance of Peano's axioms for arithmetic. We have to remind that the iterative structure of natural numbers is given to us by an intuition, essential for mathematics: the "pure intuition of iteration".

48 Our interpretation of Weyl's notion of natural numbers as a formal one can help us to explain his principle of iteration.

Explanation of Weyl's Principle of Iteration

49 In order to explain the reason why Weyl adopts the principle of iteration in his particular form, we have first to introduce the principle of substitution.

50 **Principle of substitution:**

This principle permits to use formal sets and functions to express the fact that a relation is taken as an object for another relation.
Let's present it on an example.
We will use small letters to designate variables of natural numbers and capital letters to designate variables of sets of natural numbers.

51 Let's suppose we have, for example, two relations (intensions):

$R(x, Y)$ and $S(x)$. Θ is the (formal) set obtained by the abstraction process on the relation $S(x)$. Then, the principle of substitution asserts that we can form the new relation $T(x)$ defined by: $T(x)$ *iff* $R(x, \Theta)$.

More generally, if $S(x|x_1, ..., x_n)$ is a relation (x being the only dependent variable) and if $\Theta\ (x_1, ..., x_n)$ is the (formal) function obtained by the abstraction process on the relation $S(x|x_1, ..., x_n)$, then we can form the new relation $T(x,\ x_1, ..., x_n)$ defined by:

$T(x,\ x_1, ..., x_n)$ iff $R(x, \Theta(x_1, ..., x_n))$.

52 **Principle of iteration:**

Now, we interpret Weyl's *principle of iteration* as expressing, at the logical level of the construction of relations, the completeness of every totality of sets isomorphic to the natural numbers series, that is of every totality of sets obtained from an intuitively given set X_0 by an indefinite repetition of a same homogeneous set-function.

53 Let's take, for example, a relation (intension) $R(x|Y)$. We note "$\Theta(Y)$" the function obtained by the application of the abstraction principle on the relation

"$R(x|Y)$".

We can define the "iterated" relations R^2, R^3, *etc.* by:

$$\begin{array}{lll} R^2(x,Y) & \text{iff} & R(x|\Theta(Y)), \\ R^3(x,Y) & \text{iff} & R^2(x|\Theta(Y)), \\ \quad\text{etc.} & & \end{array}$$

54 The *principle of iteration* asserts that, from the relation R and his associated (formal) function "Θ", we can form the relation $R(n,x|X)$ defined (by induction) by:

$$\begin{array}{rll} R(1,x|Y) & \text{iff} & R(x|Y) \\ \text{and } R(n+1,x|Y) & \text{iff} & R(n,x|\Theta(Y)). \end{array}$$

55 In other words, the iterated relations R^2, R^3, etc. become instances of the new relation "R^n". Quantification is allowed for natural numbers, so we can then express the relation $S(x)$ defined by: "there is a n such as $R(n, x|Y)$". This relation means that x is in a relation R with one of the sets of the series: $Y, \Theta(Y), \Theta(\Theta(Y))$, etc. which is isomorphic to the natural numbers series.

56 Therefore, Weyl's principle of iteration is a mean to construct and quantify over series of sets without betraying his predicativist position that forbids direct quantification over sets. The idea that justifies this principle is Weyl's arithmetism, *i.e.* the position according to which every totality of ideal objects isomorphic to the natural number series is complete. This idea, without being expressly formulated by Weyl, seems to be the only one that could justify his principle of iteration.

Weyl's Hierarchy of Sets

57 We have a last interpretative problem to solve. In some important passages of his text, Weyl refers to the fact that his notion of relation can be, in a way, structured in a bi-dimensional hierarchy. One of the dimensions is not problematic. Indeed, Weyl distinguishes between natural numbers, sets of natural numbers, sets of sets of natural numbers, etc. Let's call "types" these different levels. We have referred to the fact that all the variables are "typed" in Weyl's system. Variables of natural numbers will be of type 0, variables of sets of numbers will be of type 1, etc. Therefore, relations can be ordered according to the types of their variables. In particular, we have an infinity of

membership relations: $\in_1$, $\in_2$, *etc.* The first one links a number to a set of numbers. The second links a set of numbers to a set of sets of numbers, etc.

58 Nevertheless, the interpretation of the second dimension is problematic. It can't be interpreted like in Russell's Ramified Theory of Types because, if we put aside the principle of iteration, Weyl's restricted principle means that he assumes only entities of level one in Russell's hierarchy.

59 To solve this problem, we have to use the distinction we made between the abstraction process and the change of identification criterion in the transition from intension to extension. Indeed, the double-hierarchy of Weyl's sets is obtained at the logical level, before the application of the mathematical process. The possibility to define a second dimension in the classification is from the possibility of the disappearance of some set-variables, using formal sets and functions in the principle of substitution.

60 Let's define the first level of relations. Let's call it "level 0". It contains all the relations that can be defined according to Weyl's principles with the primitive entities (natural numbers and successor relation) but without using the membership relations ($\in_1$, $\in_2$, *etc.*). All these relations link together only natural numbers. They are therefore of type 0. For example: the successor relation (between two natural numbers) $f(x,y)$ itself taken as a primitive relation.

61 Let's define now the second level of relations ("level 1"). It contains all the relations that can be defined, according to Weyl's principles, with the membership relation $\in_1$ and (possibly) with the primitive entities. We can find, among those relations, relation of type 1 (that is a relation which have at least one set-variable), for example: the relation ($x \in_1 Y$) itself. Nevertheless, we can find also relation of level 1 but of type 0. For example, let's take a (formal) function $\theta(x)$ which has been obtained by the abstraction process applied to a relation $R(y|x)$ of the level 0. Then, the principle of substitution permits to form the relation $T(x, y)$ defined by: $T(x, y)$ *iff* ($x \in_1 \theta(y)$). The relation T is of type 0 (all its variables are numbers-variables) but it is of level 1 because we need the "$\in_1$" relation to define it. This possibility, at level 1, to define a 0-typed relation is what Weyl called "'recoils' to earlier levels" [Weyl 1918, 40].

62 Therefore, the greatest difference with Russell's Ramified Theory of Types is that the levels, in this hierarchy, don't depend on the supposition of the *totality* of the elements of the inferior levels. Every relation of level n depend on a finite number of relations of level $<n$. This fact permits to avoid impredicative definitions without having to relate each property or relation to one given level like in Russell's *RTT* (without the axiom of reducibility).

63 Finally, the double hierarchy evoked by Weyl must be interpreted as in Fig.:

(Double-)hierarchy of relations in *The Continuum*

ETC…				
Type 2 (There is at least one argument which refers to sets of type 2. Others arguments refer to sets of type 1 or 2 or to natural numbers)			First, we have the membership relation $X \in \Omega$ itself (Ω is a variable of type 2). We can also have relations like: $\mathbf{P(\Omega) = \forall t\, \theta(t) \in_2 \Omega}$, etc.	
Type 1 (There is at least one argument that refers to sets of type 1. Others arguments refer to sets of type 1 or to natural numbers)		First, we have the membership relation $\mathbf{x \in_1 Y}$ itself. We can also have relations like: $\mathbf{P(X) = \forall t\, (t \in_1 X)}$, $\mathbf{Q(X) = \exists s\, \exists t\, [f(s,t) \wedge (t \in_1 X)]}$, etc.	For example, $\Theta(X)$ being the formal function linked to a relation $T(y\|X)$ of level 1 and type 1, we have the relation: $\mathbf{P(X) = (X \in_2 \Theta(X))}$ etc.	
Type 0 (Every argument is linked to the category of natural numbers)	For example: the successor relation $f(x, y)$. But also relations like: $\mathbf{P(x) = \exists s\, f(s,x)}$, $\mathbf{Q(x) = \forall s\, f(s,x)}$, etc.	For example, $S(x,Y)$ being a relation of type 1 and level 1, we have: $\mathbf{P(x, y) = S(x, \theta(y))}$ where $\theta(y)$ is the formal function linked to a relation $Q(x\|y)$ of level 0, etc.	For example, $\Sigma(x)$ being the formal function linked to a relation $V(Y\|x)$ of type 1 and level 1, we have: $\mathbf{P(x) = [\theta(x) \in_2 \Sigma(x)]}$ where $\theta(x)$ is the function described in the square on the left, etc.	
	Level 0 (relations defined only from the primitive entities, *i.e.*, from the natural numbers and the successor relation *f*)	**Level 1** (relations defined only from the primitive entities and the membership relation $\in_1$)	**Level 2** (relations defined only from the primitive entities and the membership relations $\in_1$ and $\in_2$)	**ETC.**

64 The text of *The Continuum* [Weyl 1918, 29–40] suggests that Weyl tried two other possibilities before assuming this kind of bi-dimensional hierarchy (that is hierarchical only in a weak sense).

65 Firstly, he adopted a system equivalent to Russell's Ramified Theory of Types but was not satisfied. He thought that this bi-dimensional hierarchy, in a strong sense, was artificial and useless. [Weyl 1918, 32]

66 Then, he adopted a new system where we could iterate the mathematical process an indefinite number of times. In this system, we could use the "there is" principle only during the first application of the mathematical process. The other applications were restricted in the sense where we spoke of a "restricted principle" on page 5. We could then avoid impredicative definitions. Perhaps, Weyl was not satisfied of this second system because the logical level of intensions and the mathematical level of extensions were not independent. Because of the plurality of the applications of the mathematical process, we had to think of infinitely many intensional and extensional interconnected levels (like in the appendix below).

67 Only in the last system we have expounded, the logical and the mathematical levels are independent because of the uniqueness of the application of the mathematical process. In order to develop this system, Weyl had to think independently of what we called "the abstractive process" and of the change of identification criterion in what he called "the mathematical process" (see subsection "The Mathematical Process" (p. 6) and the next). In fact, his last system assumes only one application of the mathematical process but an indefinite number of applications of the abstractive process. That's why his bi-dimensional hierarchy is weak.

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APPENDIXES

Appendix - The Interrelations between The Intensional and the Extensional Levels in the Model of Constructible Sets for Set Theory

In order to explain Weyl's notions of intension and extension, we can compare them with the notions of intension and extension that are inherent to the theory of constructible sets for Set Theory. We chose this theory because it is closer to Weyl's one than the more general Set Theory.

In Set Theory, if we adopt the axiom of regularity that states that the relation ∈ on any family of sets is well-founded, we can construct the universe of sets as a cumulative hierarchy.

We define, by induction, for every ordinal number α, the class V^{α} of the sets of level α.

V is, by definition, { }. For every successor ordinal α^{+}, we have: $V^{\alpha+} = P(V^{\alpha})$ (the power-set of the class V^{α}). For every limit ordinal α, we have: $V^{\alpha} = \cap_{\beta<\alpha} V^{\beta}$ (the union of the previous classes) Then, the universe of Set Theory is thought of as the union of all the classes V^{α}. [Jech 1978, 63–65]

This definition of the universe of *Set Theory* depends on what we mean by a "subset" of a given set X (the power-set of X being the set of all subsets of X). In the universe of *constructible* sets, the power-set operator has a precise sense when it is applied to V^{α} to obtain $V^{\alpha+}$. (For constructible sets, we then call the levels of the hierarchy "L^{α}" instead of "V^{α}".) Indeed, a set Y that belongs to $L^{\alpha+}$ must admit a definition "$Y=\{x \in L^{\alpha}|\varphi(x)\}$" where φ is an intension of level α. Such an intension is a first-order formula with the "∈" symbol, the "=" symbol, constant symbols and variable symbols for sets belonging to L^{α} [Jech 1978, 175–176]. The fact that equality between sets is used in the definition of intensions and the fact that the quantifiers can be used without restriction show us that the sets that take place in these definitions are not "formal" sets like in Weyl's system but are really extensional entities. Moreover, the definition of the extensional entities that belong to $L^{\alpha+}$ depends on the intensional entities of level α which depends itself on the extensional entities that belong to L^{α}. Contrary to what happens in Weyl's system, we have an alternate series of intensional levels and of extensional levels that are interconnected.

NOTES

1. Actually, Herman Weyl doesn't use any explicit quantifier in his logical system. Nevertheless, we can express properly Weyl's ideas in the language of first-order predicate logic with which we are used to. This is what we do in the following.

2. This axiom belongs to the different set theories : *Z*, *ZF* and *ZFC*. We can't develop here the main ideas of Set Theory nor the first-order predicates calculus. The reader may refer to classical texts like [Fraenkel 1953].

3. To show the distance between Weyl's position and Set Theory, we give in the appendix an example of the interrelations between the intensional and the extensional levels in the model of the constructible sets for Set Theory.

4. We give four textual arguments that show that Weyl defends such a "formal and iterative" notion of the natural number series :

1) The fact that Weyl liken the idea of the iteration to the idea of the natural numbers series itself and of its completeness. (cf. below and [Weyl 1918, 48])
2) The particular form of the principle of iteration that Weyl assumes. We will see above that this principle expresses the idea of preceding item 1.
3) Weyl doesn't assume any isolated essence for each natural number. The only satisfactory way to define a number is to give his place in the succession.
4) Weyl agrees with an axiomatic point of view on the natural numbers series, providing that we give their real status to axioms (*i.e.*, that we take them not for a conventional definition but for an expression of the intuition of the iteration).

ABSTRACTS

In *The Continuum*, Hermann Weyl gives new bases to the notions of set and function. With them, he constructs mathematics close to physics and solves the continuum problem. Those new notions are so unusual with respect to Set Theory that they are often misunderstood.
We propose to explain the meaning of Weyl's reform of those notions. We first make a synthesis of his main epistemological thesis, and then propose a comparative approach to stress the distance between the mathematical and logical principles of *The Continuum* and those of Set Theory. Our discussion will be centred on the distinction between intension and extension, and on the place Weyl gives to natural numbers for the foundations of analysis.

Dans le *Continu*, Hermann Weyl donne une nouvelle assise aux notions d'ensemble et de fonction, pour assurer aux mathématiques leur applicabilité à la physique, et résoudre ainsi le problème du continu. Les notions introduites, éloignées de la théorie des ensembles, prêtent à confusion et à multiples interprétations.
Nous nous proposons d'éclairer le sens du déplacement que Weyl opère dans ces notions. Nous présentons une synthèse des thèses épistémologiques soutenues dans *Le Continu* et résolvons certains problèmes interprétatifs. Par une approche comparative, nous soulignons l'écart entre les principes logico-mathématiques du *Continu* et ceux de la théorie des ensembles. Nous nous centrons sur la distinction entre intension et extension, et sur la place attribuée aux entiers naturels pour le fondement des mathématiques.

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